UNIVERSITY OF GLASGOW

DEPARTMENT OF AERONAUTICS AND FLUID MECHANICS

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SUMMARY

The structural response of filament-wound cylindrical shells to a combination of applied loadings is investigated. In many cases the membrane theory forms an adequate basis for the analysis. By suitable arrangement of the filament orientations and layer thicknesses, an efficient structure can be achieved. Numerical examples are given to show their influence on the stress components. The strength of the shells can be judged according to some criterion. In the Appendix, the determination of the elastic constants is given.

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INTRODUCTION

During recent years considerable interest has been shown in the development of filament reinforced materials by virtue of the fact that such materials, apart from their many other merits, possess a very high strength to density ratio. Normally a soft material is used as a binder in which much stronger fibres are embedded. Thus the fibres become the major load-carrying elements, and the binding matrix, though it may also provide a certain load-carrying capacity and stiffness, acts as a load transmitting medium. Great strength becomes possible if continuous fibres are employed, as the stress concentrations which would inevitably be set up at the end of a fibre, can thus be avoided. The efficiency of a filamentary structure can be further increased by the appropriate orientation of the fibres along the direction in which the maximum strength is required. In structures where it is not possible to orient the fibres exactly in the direction of the principal stress, they may be arranged to orient at some angle thereto. Sometimes it is necessary to have different angles of winding between layers in order to achieve a balanced structure. Filament-wound pressure vessels for space applications have been shown to be highly efficient structures, but their potential as structural members for general applications has not been widely appreciated. In the field of commercial products, Zoller [1] lists a number of possibilities for the application of fibre-glass filaments impregnated with epoxy resin. The purpose of this paper is to provide some understanding of the structural behaviour of filament wound cylindrical shells.

NOTATIONS

x, y, z	coordinates as shown in Fig. 1.
u, v, w	displacements as shown in Fig. 1.
l, m	direction cosines.
r	radius of cylinder.
t	thickness.
p	pressure.
v_f	fibre volume fraction.
N	resultant force per unit length.
M	resultant moment per unit length.
L	length of cylinder.
E	elastic modulus.
G	shear modulus.
$\bar{X}$, $\bar{Y}$, $\bar{Z}$	tensile yield stresses in the principal directions.
S	yield stress in shear.
C	filament contiguity factor.
n	number of layers.
$[\sigma]$	stress matrix
$[\epsilon]$	strain matrix
$[A]$	transformed compliance matrix as defined by equ. 13.
$[B]$	flexibility matrix.
$[C]$	stiffness matrix.
$[L]$	stress transformation matrix.
$[H]$	strain transformation matrix.
$[N]$	resultant force matrix.
$[M]$	resultant moment matrix.
$[A^*]$	as defined by equation 17.
$[K]$	as defined by equation A-12.

$\left[A^{xx}\right]$	as defined by equation 19.
α	orientation of filament with reference to x-axis.
σ_x, σ_θ	direct stress.
$\sigma_{x\theta}$	shear stress.
$\epsilon_x, \epsilon_\theta$	direct strain.
$\epsilon_{x\theta}$	shear strain.
δ	distance of layer from coordinate face.
ζ	thickness ratio.
ξ	non-dimensional stress component.
χ	change of curvature.
ν	Poisson's ratio.
β_I, β $\mu_1, \mu_2, \mu_I,$ ϕ_I, ϕ_{II}	non-dimensional parameters denoting modulus ratios.
η	coefficient of mutual influence.

Subscripts

x, y	x and y components.
θ	circumferential component.
I, II	components in the principal directions of a helical layer.
L, T	components in the longitudinal and transverse direction of a unidirectional layer.
α	layer with filament angle α.
o, e	odd and even layers.
c, h	circumferential and helical layers.
i	ith layer.
f	fibre.

STRUCTURAL TREATMENT AND BASIC EQUATIONS

Although the fibres and the binding matrix are normally taken as elastic, isotropic and homogeneous, nevertheless because of the orientation of the fibres, the composite material becomes anisotropic. The strength of the structural member in a given direction therefore depends on the properties and distribution of the fibre and the matrix as well as on the orientation of the fibres. Among the surfaces of revolution convenient for filament winding, circular cylinders are the most common form. Along the cylinders, winding can be circumferential, longitudinal and helical as shown in Fig. 1. In the helical winding, a complete layer is interspersed with fibres oriented at angle $+\alpha$ and $-\alpha$ with the axis of the cylinder. These two interlocking half layers may be treated as a single orthotropic layer with their principal directions coinciding with the axial and circumferential directions of the cylinder.

A laminated composite consisting of a number of multi-phase layers is in general anisotropic and heterogeneous. With filament winding, however, the composite is usually binary and each layer, be it circumferentially, longitudinally or helically wound, can be considered as orthotropic. When these layers are bonded together, heterogeniety occurs only between layers so that within each layer homogeneity can be assumed and hence the stiffness coefficients are continuous. The response of the structure to the application of external loads is conveniently described by the stiffness coefficients according to the governing equations of the generalised Hooke's law

$$[\sigma] = [C]\,[\epsilon] \qquad (1)$$

where $[C]$ is the stiffness matrix which in general contains 36 constants. According to the classical theory of thin shells, both the stress normal to the plane and the transverse shear deformations in the planes of the normal sections can be neglected. Owing to the plane of elastic symmetry, it is possible to reduce the number of constants to 6. The stress-strain relations of the ith layer become

$$\begin{bmatrix} \sigma_x \\ \sigma_\theta \\ \sigma_{x\theta} \end{bmatrix}_i = \begin{bmatrix} C_{11} & C_{12} & C_{16} \\ C_{12} & C_{22} & C_{26} \\ C_{16} & C_{26} & C_{66} \end{bmatrix}_i \begin{bmatrix} \epsilon_x \\ \epsilon_\theta \\ \epsilon_{x\theta} \end{bmatrix}_i \qquad (2)$$

With these coefficients we can obtain the resultant forces (N) per unit length and the resultant moments (M) per unit length from the statically equivalent internal forces and moments.[2] Assuming that the wall thickness of the shell is very small compared with its radius, we obtain

$$N = \sum_{i=1}^{n} \int \sigma_i \, dz$$
$$M = \sum_{i=1}^{n} \int \sigma_i \, z \, dz \qquad (3)$$

where the integral extends across the thickness of the ith layer and the summation is for the n layers.

For a cylindrical shell, the strain displacement relationships are given by the following expressions :-

$$\begin{aligned} \varepsilon_x &= \varepsilon_x^\circ - z\chi_x \\ \varepsilon_\theta &= \varepsilon_\theta^\circ - z\chi_\theta \\ \varepsilon_{x\theta} &= \varepsilon_{x\theta}^\circ - z\chi_{x\theta} \end{aligned} \qquad (4)$$

where

$$\epsilon_x^{\,o} = \frac{\partial u}{\partial x}$$

$$\epsilon_\theta^{\,o} = \frac{1}{r}\left(\frac{\partial v}{\partial \theta} - \omega\right)$$

$$\epsilon_{x\theta}^{\,o} = \frac{\partial v}{\partial x} + \frac{1}{r}\frac{\partial u}{\partial \theta}$$

are the strains at the middle surface of the shell and u, v, ω, are the corresponding displacements and

$$\chi_x = \frac{\partial^2 \omega}{\partial x^2}$$

$$\chi_\theta = \frac{1}{r^2}\left(\frac{\partial v}{\partial \theta} + \frac{\partial^2 \omega}{\partial \theta^2}\right)$$

$$\chi_{x\theta} = \frac{2}{r}\left(\frac{1}{2}\frac{\partial v}{\partial x} + \frac{\partial^2 \omega}{\partial x \partial \theta}\right)$$

are the changes of curvature.

The stress components in the ith layer can be obtained from equations (2) and (4). That is

$$[\sigma]_i = [C]_i\,[\epsilon^o]_i - z\,[C]_i\,[\chi]_i \qquad (5)$$

For homogeneous layers, according to the earlier statement, $[C]$ is independent of z and equation (3) becomes

$$[N] = \sum_{i=1}^{n}(\delta_i - \delta_{i-1})[C]_i[\epsilon^o]_i - \sum_{i=1}^{n}\frac{1}{2}(\delta_i^2 - \delta_{i-1}^2)[C]_i[\chi]_i$$

$$[M] = \frac{1}{2}\sum_{i=1}^{n}(\delta_i^2 - \delta_{i-1}^2)[C]\,[\epsilon^o] - \sum_{i=1}^{n}\frac{1}{3}(\delta_i^3 - \delta_{i-1}^3)[C]_i[\chi]_i \qquad (6)$$

where δ_i is the distance measured from the coordinate surface to the upper face of the ith layer taking the appropriate sign according to the direction of the z-axis.

Equation (6) represents the elasticity relationships which, when used in conjunction with the fundamental equilibrium equations or the energy equation with the appropriate boundary conditions, make possible the solution of a laminated, anisotropic, quasi-homogeneous, cylindrical shell.

MEMBRANE THEORY

In many thin shell problems, the deformations are such that the stresses in the shells are mainly due to the middle surfaces forces (N) and the stresses due to bending and twisting moments (M) are very small. These conditions, as applied to the case of isotropic shells [3], are also valid for thin anisotropic laminated shells, provided that the layers are symmetrically laid about the middle surfaces. Since the wall of a filament wound cylinder is usually made up of numerous layers with alternate patterns, the symmetry assumption can still give reasonably good results whether there are odd layers or even layers. In this case, the problem of stress analysis is greatly simplified. Equation (6) thus reduces to

$$\left[N\right] = \sum_{i=1}^{n} (\delta_i - \delta_{i-1}) \left[C\right]_i \left[\epsilon^o\right]_i \qquad (7)$$

and the total strains $\left[\epsilon\right]_i$ can be taken as the strains in the middle surface in the place of $\left[\epsilon^o\right]_i$. This is equivalent to assuming a uniform distribution of stresses applied over the thickness of the shell, i.e.,

$$N = t\sigma \qquad (8)$$

where t is the wall total thickness. Equation (7) can be written as

$$t\left[\sigma\right] = \sum_{i=1}^{n} t_i \left[\sigma\right]_i \qquad (9)$$

where t_i is the thickness of the ith layer and $\left[\sigma\right]_i = \left[C\right]_i \left[\epsilon\right]_i$ takes the same form as equation (1).

CYLINDRICAL SHELLS CONSISTING OF ORTHOTROPIC LAYERS

The shell is considered to consist of n orthotropic unidirectional layers alternately laid with the filaments of the odd layers oriented at an angle α_o and the even layers α_e with the x-axis, as shown in Fig. 2. Let the total thickness of the odd layers be t_o and that of the even layers t_e. Although for the normal technique of continuous winding, the arrangement of alternate unidirectional layers with arbitrary filament orientations may not be adopted in practice, the general formulation procedure can be easily extended to cover shells with any combination of orthotropic layers.

Let us first consider one unidirectional layer with filaments orienting at an angle α with the x-axis. The stress and strain transformations between the principal directions L and T of the layer and the x and y directions are given by

$$\begin{aligned} [\sigma]_\alpha &= [L]_\alpha [\sigma] \\ [\epsilon]_\alpha &= [H]_\alpha [\epsilon] \end{aligned} \tag{10}$$

where $[L]_\alpha$ is the stress transformation matrix and $[H]_\alpha$ is the strain transformation matrix and they are

$$[L]_\alpha = \begin{bmatrix} \ell & m^2 & 2m\ell \\ m^2 & \ell^2 & -2m\ell \\ -m\ell & \ell m & \ell^2 - m^2 \end{bmatrix}$$

$$[H]_\alpha = \begin{bmatrix} \ell^2 & m^2 & m\ell \\ m^2 & \ell^2 & -m\ell \\ -2\ell m & 2m\ell & \ell^2 - m^2 \end{bmatrix}$$

where ℓ and m are the direction cosines with $\ell = \cos\alpha$ and $m = \sin\alpha$. The relationships between the stress and strain components are

$$[\epsilon]_\alpha = [B][\sigma]_\alpha \tag{11}$$

Since the layer has orthotropic elastic properties, the flexibility matrix is given by

$$[B] = \begin{bmatrix} \frac{1}{E_L} & -\frac{\nu_{LT}}{E_T} & 0 \\ -\frac{\nu_{TL}}{E_L} & \frac{1}{E_T} & 0 \\ 0 & 0 & \frac{1}{G_{LT}} \end{bmatrix}$$

Many investigators have attempted the determination of the elastic constants and their work has resulted in many publications. Only a few are quoted here (4 - 9). More details are given in the Appendix.

Combining equations (10) and (11), we obtain

$$\begin{aligned} [\epsilon] &= [H]_\alpha^{-1} [B] [L]_\alpha [\sigma] \\ &= [A]_\alpha [\sigma] \end{aligned} \qquad (12)$$

Since the inverse matrix $[H]_\alpha^{-1}$ is in fact equal to the transpose matrix $[L]_\alpha'$, we have

$$[A]_\alpha = [L]_\alpha' [B] [L]_\alpha \qquad (13)$$

From equation (13), the elastic properties in the x,y directions can be determined in terms of the elastic properties in the L, T directions. Using the engineering constants we obtain the well known expressions

$$A_{11} = \frac{1}{E_x} = \frac{1}{E_L}\left[\ell^2 + \left(\frac{1}{\mu_1} - 2\nu_{TL}\right)\ell^2 m^2 + \frac{m^4}{\beta}\right]$$

$$A_{22} = \frac{1}{E_y} = \frac{1}{E_L}\left[m^4 + \left(\frac{1}{\mu_1} - 2\nu_{TL}\right)\ell^2 m^2 + \frac{\ell^4}{\beta}\right]$$

$$A_{66} = \frac{1}{G_{xy}} = \frac{1}{G_{LT}}\left[1 + (\mu_1 + \mu_2 + 2\mu_1\nu_{TL} - 1)\,4\ell^2 m^2\right]$$

$$A_{12} = -\frac{\nu_{yx}}{E_x} = \frac{1}{E_L}\left[\left(1 + \frac{1}{\beta} + 2\nu_{TL} - \frac{1}{\mu_1}\right)\ell^2 m^2 - \nu_{TL}\right] \tag{14}$$

$$A_{26} = -\frac{\eta_{y,xy}}{G_{xy}} = -\frac{\eta_{xy,y}}{E_y} = \frac{\ell m}{E_L}\left[2\left(\frac{\ell^2}{\beta} - m^2\right) - \left(\frac{1}{\mu_1} - 2\nu_{TL}\right)(\ell^2 - m^2)\right]$$

$$A_{16} = -\frac{\eta_{x,xy}}{G_{xy}} = -\frac{\eta_{xy,x}}{E_x} = \frac{\ell m}{E_L}\left[2\left(\frac{m^2}{\beta} - \ell^2\right) + \left(\frac{1}{\mu_1} - 2\nu_{TL}\right)(\ell^2 - m^2)\right]$$

where the dimensionless parameters

$$\mu_1 = \frac{G_{LT}}{E_L}, \quad \mu_2 = \frac{G_{LT}}{E_T}, \quad \beta = \frac{E_T}{E_L},$$

$\eta_{x,xy}$ and $\eta_{y,xy}$

are known as the coefficients of mutual influence. $\eta_{x,xy}$ characterises the elongation in the x direction due to a shear stress acting in the $x - y$ plane and $\eta_{xy,x}$ characterises a shear strain in the $x - y$ plane due to a normal stress in the x direction. ν_{yx} is the Poisson's ratio signifying an elongation in the y direction due to a normal stress in the x direction.

Now when the unidirectional layers are alternately bonded together with orientations α_o and α_e, equation (9) gives

$$t_o\,[\sigma]_o + t_e\,[\sigma]_e = t\,[\sigma] \tag{15}$$

where $[\sigma]$ represents the applied stress components.

The requirements of consistent deformations demand that

$$[\epsilon]_o = [\epsilon]_e = [\epsilon] \tag{16}$$

Using the appropriate stres-strain relationship (equation 12) for the odd and even layers denoted by subscripts $_o$ and $_e$, respectively, we find the stress components of the odd and even layers in the x and y directions from equations (15) and (16). They are

$$\begin{aligned} [\sigma]_o &= [A^*]^{-1}[A]_{\alpha e}[\sigma] \\ [\sigma]_e &= [A^*]^{-1}[A]_{\alpha o}[\sigma] \end{aligned} \tag{17}$$

where $[A^*] = \frac{1}{t}\left\{[A]_{\alpha o}\, t_e + [A]_{\alpha e}\, t_o\right\}$

and $[A]_{\alpha o}$ and $[A]_{\alpha e}$ are the flexibility matrices according to equation (13).

In the study of failure of a laminate, it is necessary to establish first the states of stress and strain in each layer. The investigation of its strength can be conveniently carried out if the stresses are resolved along the principal directions of the layer. For unidirectional layers, the stresses along the principal directions are obtained by simple transformations, i.e.,

$$\begin{aligned} [\sigma]_{\alpha o} &= [L]_{\alpha o}[A^*]^{-1}[A]_{\alpha e}[\sigma] \\ [\sigma]_{\alpha e} &= [L]_{\alpha e}[A^*]^{-1}[A]_{\alpha o}[\sigma] \end{aligned} \tag{18}$$

The application of equation (17) need not be restricted to unidirectional layers, but to orthotropic layers in general. In fact, the same expression can be extended to cover more than two alternate layers. For a symmetrically laminated shell of n different orthotropic layers (all the identical layers are grouped into one layer), the stress matrix of the ith layer takes the form

$$[\sigma]_i = [A]_i^{-1} [A^{**}]^{-1} [\sigma] \tag{19}$$

where $\quad [A^{**}] = \frac{1}{t} \sum_{k=1}^{n} (t_k [A]_k^{-1})$

Circumferential and longitudinal windings

The combination of circumferential and longitudinal winding has often been employed for pressure vessels largely because the filaments in the longitudinal direction can be easily extended to form end domes of the vessel. The filaments in the circumferential direction are relied upon to provide the hoop strength. The angle of the longitudinal filaments depends on the size of the polar openings but the circumferential filaments often lie very close to the y-axis. For our purpose it suffices to assume $\alpha_o = 0^\circ$ and $\alpha_e = 90^\circ$. Let the ratio of the total thickness of the odd layers to the total thickness of the even layers be ζ. Let also the stress components be changed into dimensionless parameters by dividing each by σ_θ, so that

$$\xi_x = \sigma_x/\sigma_\theta, \qquad \xi_\theta = 1 \text{ and } \xi_{x\theta} = \sigma_{x\theta}/\sigma_\theta.$$

Then after some computation, equation 17 appears as

$$\begin{bmatrix} \xi_x \\ \xi_\theta \\ \xi_{x\theta} \end{bmatrix}_o = \frac{1}{(\beta+\zeta)(1+\beta\zeta)-\nu_{TL}^2(1+\zeta)^2\beta^2} \begin{bmatrix} \{(1-\nu_{TL}^2\beta^2)+(1+\beta-\nu_{TL}^2\beta^2 2)\zeta+\beta(1-\nu_{TL}^2\beta)\zeta^2\}\xi_x + \nu_{TL}\beta(\beta-1)(1+\zeta \\ \nu_{TL}\beta(1-\beta)(1+\zeta)\xi_x + \{\beta^2(1-\nu_{TL}^2)+\beta(1+\beta-2\beta\nu_{TL}^2)\zeta+\beta(1-\beta\nu_{TL}^2)\zeta^2\} \\ \{(\beta+\zeta)(1+\beta\zeta)-\nu_{TL}^2(1+\zeta)^2\beta^2\}\xi_{x\theta} \end{bmatrix}$$

$$\begin{bmatrix} \xi_x \\ \\ \xi_\theta \\ \\ \xi_{x\theta} \end{bmatrix}_e = \frac{1}{(\beta+\zeta)(1+\beta\zeta)-\nu_{TL}^2(1+\zeta^2)\beta^2} \begin{bmatrix} \left\{\beta(1-\nu_{TL}^2\beta)+\beta(1+\beta-2\beta\nu_{TL}^2)\zeta+\beta^2(1-\nu_{TL}^2)\zeta^2\right\}\xi_x+\left\{\nu_{TL}\beta(1-\beta)\zeta+\nu_{TL}\beta(1-\beta)\zeta^2\right\} \\ \\ \left\{\beta\nu_{TL}(\beta-1)\zeta+\beta\nu_{TL}(\beta-1)\zeta^2\right\}\xi_x+\left\{\beta(1-\beta\nu_{TL}^2)+(1+\beta-2\beta^2\nu_{TL}^2)\zeta+(1-\beta^2\nu_{TL}^2)\zeta^2\right\} \\ \\ \left\{(\beta+\zeta)(1+\beta\zeta)-\nu_{TL}^2(1+\zeta)^2\beta^2\right\}\xi_{x\theta} \end{bmatrix}$$

$$\text{where} \quad \beta = \frac{E_T}{E_L} \tag{20}$$

Owing to the orthotropic characteristics, the applied shear stress does not influence the direct stress components neither do the applied direct stresses influence the shear stress component.

The average values of elastic moduli of glass-epoxy resins are used here for numerical examples.

$E_f = 10.5 \times 10^6$ psi

$E_m = 0.5 \times 10^6$ psi

$\nu_f = 0.2$

$\nu_m = 0.35$

Assuming the fibre volume fraction to be 0.7, we obtain the following constants from Fig. 7-9 from curves with C = 0.2

$E_L = 6.75 \times 10^6$ psi

$E_T = 2.4 \times 10^6$ psi

$G_{LT} = 1 \times 10^6$ psi

$\nu_{TL} = 0.245$

The direct stress components in the odd and even layers are plotted in Fig. 3 for varying ζ and ξ_x. It is interesting to see that as ξ_x

is increased to indicate a greater longitudinal load compared with the hoop load, ξ has to be increased in order to bring the stress components (i.e. $(\xi_x)_o$ and $(\xi_\theta)_e$) near to each other. When $(\xi_x)_o = (\xi_\theta)_e$, we also find $(\xi_\theta)_o = (\xi_x)_e$. Under these conditions, it is said that the structure is a balanced structure. At $\xi_x = 0.5$, a case representing the application of internal pressure alone, the balanced structure demands the thickness ratio to be $\xi = 0.118$. According to the netting analysis, however, in which the resin is assumed to have zero load carrying capacity, the prediction would give the thickness ratio to be $\xi = 0.5$. The error made by this oversimplified analysis is very obvious.

By equating $(\xi_x)_o$ to $(\xi_\theta)_e$ the optimum thickness ratio for a given load ratio ξ_x can be obtained. This is shown in Fig. 4. The limiting values of ξ_x are 0.4 and 2.5, beyond which no balanced structure is possible. In designing a filament wound structure, the orientation of the fibres is so arranged as to give the best structural efficiency in resisting a given design loading system. If the loads are applied separately, it is likely that the same efficiency can no longer be maintained. Under such circumstances, the whole sequence of the application of loads should be considered and curves such as those shown in Fig. 3 would become useful design charts.

<u>Circumferential and helical windings</u>

The advantages of using helical winding are two-fold. One is that by varying the helical angle we can vary the circumferential to longitudinal strength ratio without necessarily altering the thickness ratio of the layers. The other is that a helical layer provides a higher shearing strength than an unidirectional layer and this is very useful in a structure in which a high shear strength is required. There appears to be no great benefit from having layers consisting of different helical angles except in cases where ξ_x varies with location or during the working period and in such cases the use of various helical layers is necessary to ensure a good overall strength. The common practice, however, is to reinforce the helical layers with circumferential layers. Since both the circumferential layers and the helical layers are orthotropic and if layer-symmetry is assumed, the internal stress components can be determined from equation (19) in which the necessary elastic constants are obtained from the Appendix. Assume the same angle for all helical layers and let the total thickness of these layers be t_h and the total thickness of the circumferential layers be t_c. Then $\mathscr{S} = t_c/t_h$. The flexibility matrices are

$$[A]_c = \frac{1}{E_L}\begin{bmatrix} \frac{1}{\beta} & -\nu_{TL} & 0 \\ -\nu_{TL} & 1 & 0 \\ 0 & 0 & \frac{1}{\mu_1} \end{bmatrix}$$

and

$$[A]_h = \frac{1}{E_I}\begin{bmatrix} 1 & -\nu_{II\,I} & 0 \\ -\nu_{II\,I} & \frac{1}{\beta_I} & 0 \\ 0 & 0 & \frac{1}{\mu_I} \end{bmatrix}$$

where c and h denote circumferential and helical respectively.

Using the nondimensional parameters such as

$$\beta = \frac{E_T}{E_L}, \quad \beta_I = \frac{E_{II}}{E_I}, \quad \phi_I = \frac{E_L}{E_I}, \quad \phi_{II} = \frac{E_L}{E_{II}}, \quad \mu_1 = \frac{G_{LT}}{E_L} \text{ and } \mu_I = \frac{G_{I\,II}}{E_I}$$

then the nondimensional stress components of the two different group of layers are given in the following expressions.

$$\begin{bmatrix}\xi_x\\ \xi_\theta\\ \xi_{x\theta}\end{bmatrix}_c=\frac{\phi_I(1+\zeta)}{\left(\frac{1}{\beta}+\zeta\phi_I\right)(1+\zeta\phi_{II})-(\nu_{TL}+\zeta\phi_I\nu_{LT})^2}\begin{bmatrix}\left\{(1+\zeta\phi_{II})-\nu_{LT}(\nu_{TL}+\zeta\phi_I\nu_{LT})\right\}\xi_x-\left\{\nu_{LT}(1+\zeta\phi_{II})-(\nu_{TL}+\zeta\phi_I\nu_{LT})\frac{1}{\beta_I}\right\}\\ \left\{(\nu_{TL}+\zeta\phi_I\nu_{LT})-\nu_{LT}\left(\frac{1}{\beta}+\zeta\phi_I\right)\right\}\xi_x-\left\{\nu_{LT}(\nu_{TL}+\zeta\phi_I\nu_{LT})-\frac{1}{\beta_I}\left(\frac{1}{\beta}+\zeta\phi_I\right)\right\}\\ \left\{\frac{\left(\frac{1}{\beta}+\zeta\phi_I\right)(1+\zeta\phi_{II})-(\nu_{TL}+\zeta\phi_I\nu_{LT})^2}{\mu_I\left(\frac{1}{\mu_1}+\zeta\phi_I\frac{1}{\mu_I}\right)}\right\}\xi_{x\theta}\end{bmatrix}$$

$$\begin{bmatrix}\xi_x\\ \xi_\theta\\ \xi_{x\theta}\end{bmatrix}_h=\frac{(1+\zeta)}{\left(\frac{1}{\beta}+\zeta\phi_I\right)(1+\zeta\phi_{II})-(\nu_{TL}+\zeta\phi_I\nu_{LT})^2}\begin{bmatrix}\left\{\frac{1}{\beta}(1+\zeta\phi_{II})-\nu_{LT}(\nu_{TL}+\zeta\phi_I\nu_{LT})\right\}\xi_x-\left\{\nu_{TL}(1+\zeta\phi_{II})-(\nu_{TL}+\zeta\phi_I\nu_{LT})\right\}\\ \left\{(\nu_{TL}+\zeta\phi_I\nu_{LT})\frac{1}{\beta}-\nu_{TL}\left(\frac{1}{\beta}+\zeta\phi_I\right)\right\}\xi_x-\left\{\nu_{TL}(\nu_{TL}+\zeta\phi_I\nu_{LT})-\left(\frac{1}{\beta}+\zeta\phi_I\right)\right\}\\ \left\{\frac{\left(\frac{1}{\beta}+\zeta\phi_I\right)(1+\zeta\phi_{II})-(\nu_{TL}+\zeta\phi_I\nu_{LT})^2}{\mu_1\left(\frac{1}{\mu_1}+\zeta\phi_I\frac{1}{\mu_I}\right)}\right\}\xi_{x\theta}\end{bmatrix} \tag{21}$$

Using the same material composition as in the previous section and taking the appropriate elastic constants from Fig. 7-10, the nondimensional direct stress components in different layers are computed for various helical angles under four different applied stress ratios ξ_x. These values are plotted in Fig. 5. The thickness ratio in all cases is taken as 0.2. Although it is possible to find an optimum value of ζ within a certain load range if the helical angle is fixed, it is thought to be more interesting to see how the stress components are influenced by the helical angle. Take the ξ_x = 0.1 case for example. Owing to a high Poisson's ratio in the helical layer, compressive stress in the longitudinal direction in the circumferential layer is set up, although the structure is subjected to tensile loads in all directions. As ξ_x increases in value, a reduction in the helical angle makes the structure progressively more efficient as is shown by the curves 1 and 4. The shear stress components are plotted in Fig. 6. As expected the helical layers become more efficient in carrying the shear load when the helical angle approaches 45°.

STRENGTH CRITERION

In designing a filament wound structure, a knowledge of its strength is desirable. From an engineering point of view, failure implies that the structure has been badly deformed by the applied loads to such an extent that it can no longer perform its function properly. A laminated material may fail through the failure of one of its layers. As shown in Fig. 4-6, the stresses in an individual layer may differ from the apparent stresses obtained by assuming the applied load to be divided by the thickness of the shell wall. When the stresses and strains have been calculated, the failure of the structure will have to be judged according to some strength criterion. Various criteria of failure of metals have been introduced. Among the best known are the maximum shear stress criterion, the maximum shear strain energy criterion and the maximum principal stress criterion (10). The first two are used for ductile materials whilst the last is applicable to more brittle materials. In the case of glass-epoxy resin composite, glass is brittle and yet the resin is ductile. There is not enough experimental evidence available to show which is in better agreement with reality. A yield criterion for an anisotropic metal was postulated by Hill (11). For a thin sheet in a state of plane stress, his generalised equation reduces to

$$\left(\frac{\sigma_x}{\bar{X}}\right)^2 - 2H\sigma_x\sigma_y + \left(\frac{\sigma_y}{\bar{Y}}\right)^2 + \left(\frac{\sigma_{xy}}{S}\right)^2 = 1 \qquad (22)$$

where $H = \frac{1}{2}\left(\frac{1}{\bar{X}^2} + \frac{1}{\bar{Y}^2} + \frac{1}{\bar{Z}^2}\right)$

$\bar{X}$, $\bar{Y}$ and $\bar{Z}$ are the tensile yield stresses in the principal directions and S is the yield stress in shear in the x-y plane while σ_x, σ_y and σ_{xy} are the applied stresses. For composite material Tsai (12) proposed that the following equation could be used

$$\left(\frac{\sigma_x}{\bar{X}}\right)^2 - \left(\frac{\sigma_x\sigma_y}{\bar{Y}^2}\right) + \left(\frac{\sigma_y}{\bar{Y}}\right)^2 + \left(\frac{\sigma_{xy}}{S}\right)^2 = 1 \qquad (23)$$

An ellipsoidal expression was also suggested in reference (13) for wood aircraft structural design.

$$\left(\frac{\sigma_x}{\sigma_{xf}}\right)^2 + \left(\frac{\sigma_y}{\sigma_{yf}}\right)^2 + \left(\frac{\sigma_{xy}}{\sigma_{xyf}}\right)^2 = 1 \qquad (24)$$

where σ_{xf}, σ_{yf} and σ_{xyf} are the longitudinal, transverse and shear stresses respectively at failure in the absence of other stresses. Extended experimental investigations will be required before it can be decided which criterion is a reliable one to use. Equation (24), however, will predict lower values of failure if both σ_x and σ_y are in tension and therefore it is safer to use from the designing point of view. If either one or both are in compression the structure may fail in instability.

CONCLUSION

The abilityof filament wound cylindrical shells to resist internal pressure and other loadings has been shown. With suitable layer thickness ratio and appropriate filament orientations, the structure can be expected to function efficiently. Although numerical analysis has been carried out on the basis of the membrane theory, the general equations can be applied to shells where resultant moments may exist. This requires the determination of the responsive matrices (equation 3) involving the stiffness matrix of each individual layer according to its location with reference to the coordinate surface. The membrane theory, however, should give a reasonable basis for the analysis of shells with high radius to thickness ratio and consisting of a reasonable number of layers if they were not symmetrically laid. The stress system thus obtained will enable failure to be predicted according to the strength criterion suggested.

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APPENDIX

DETERMINATION OF ELASTIC CONSTANTS

In a laminated composite material, the layers may be made of the same or different materials. The elastic properties may then differ from layer to layer. With a filament wound structure, the layers are composed of binary constituents and are orthotropic in character. The elastic properties therefore vary mainly with the orientation of the filaments. When the material is orthotropic, no shear strain is produced if the normal stresses are applied along the principal directions. Likewise, a shear stress when applied parallel and perpendicular to the principal axes, gives rise to no direct strains in these directions. Knowing the properties along the principal axes, properties along any other directions can be obtained. The elastic properties along the principal directions of each layer therefore constitute the basic properties of a laminated shell in its response to external loads.

Elastic constants of a unidirectional layer

A unidirectional layer is that which consists of one single orientation of the fibres which are assumed to be uniformly distributed in the binding matrix. These fibres are assumed to be circular in cross section, continuous and isotropic and have known elastic properties. The binding matrix is also assumed isotropic and has known elastic properties.

Longitudinal modulus E_L

The modulus along the direction of the fibres is found to follow very closely the law of mixtures in which both constituent elements contribute to the overall stiffness of the material in proportion to their individual stiffness and volume fraction. Accordingly, we have

$$E_L = E_f v_f + (1 - v_f) E_m \qquad \text{A - 1}$$

where E_f and E_m are the moduli of elasticity of fibre and matrix respectively, and v_f is the volume fraction of the fibre.

Poisson's ratios

The major Poisson's ratio, which characterises an elongation in the transverse direction caused by a normal stress in the longitudinal direction, is also found to follow the law of mixture, i.e.

$$\nu_{TL} = \nu_f v_f + (1 - v_f) \nu_m \qquad \text{A - 2}$$

where ν_f and ν_m are the Poisson's ratios of fibre and matrix respectively.

The minor Poisson's ratio can be obtained by applying the Maxwell's reciprocal theorem, i.e.

$$\nu_{LT} = \frac{\nu_{TL} E_T}{E_L} \qquad \text{A - 3}$$

Both E_L and ν_{TL} for glass-epoxy resin composite are plotted in Fig. 7.

Transverse modulus E_T

The determination of the elastic modulus normal to the direction of the fibres has attracted the attention of many investigators. The majority of those quoted here (4 - 9) based their analysis on the theorems of minimum potential energy. Their results are given below except that of Rosen and Hashin which require the use of digital computers.

Tsai's result:

$$E_T = 2\left[1 - \nu_m - (\nu_f - \nu_m) v_f\right]\left[(1-C)\frac{K_f(2K_m + G_m) - G_m(K_f - K_m)(1 - v_f)}{(2K_m + G_m) + 2(K_f - K_m)(1 - v_f)} + C\frac{K_f(2K_m + G_f) + G_f(K_m - K_f)(1 - v_f)}{(2K_m + G_f) + (K_f - K_m)(1 - v_f)}\right] \qquad \text{A - 4}$$

where

$$G_f = \frac{E_f}{2(1+\nu_f)}, \quad G_m = \frac{E_m}{2(1+\nu_m)}, \quad K_f = \frac{E_f}{2(1-\nu_f)}, \quad K_m = \frac{E_m}{2(1-\nu_m)} .$$

C is a factor indicating the degree of contiguity of the fibres. For isolated fibres $C=0$; for contiguous fibres, $C=1$.

Shaffer's result:

$$E_T = E_m\left\{1-\left(1-\frac{E_m}{E_f}\right)\left[0.8247\,(v_f)^{\frac{1}{2}} - v_f\right]\right\}\left[1-0.8247\,(v_f)^{\frac{1}{2}}\left(1-\frac{E_m}{E_f}\right)\right] \qquad \text{for } v_f \le 0.68$$

$$E_T = E_m\Big/\left[1 - v_f\left(1-\frac{E_m}{E_f}\right)\right] \qquad \text{for } v_f > 0.68$$

A - 5

Whitney and Riley's result:

$$E_T = \frac{2E_L(1-\nu_{TL})\left[(K_f'+G_m)K_m' - (K_f'-K_m')G_m v_f\right]}{E_L\left[(K_f'+G_m)-(K_f'-K_m')v_f\right] + 4\nu_{TL}^2\left[(K_f'+G_m)K_m - (K_f'-K_m')G_m v_f\right]} \qquad \text{A - 6}$$

where

$$K_f' = \frac{E_f}{2(1-\nu_f-2\nu_f^2)}, \quad K_m' = \frac{E_m}{2(1-\nu_m-2\nu_m^2)}.$$

These results based on the properties of glass and epoxy resin are plotted in Fig. 8. Experimental results of Tsai and Card indicated that Tsai's curve with $C=0.2$ gives good agreement. This is adopted for calculations in the paper.

Shear Modulus G_{LT}

The shear modulus derived by Rosen and Hashin incorporated with Tsai's factor of contiguity C is given by

$$G_{LT} = (1-C)\frac{[(G_f+G_m)+(G_f-G_m)v_f]}{(G_f+G_m)-(G_f-G_m)v_f}G_m + C\frac{2G_m+(G_f-G_m)v_f}{2G_f-(G_f-G_m)v_f}G_f \qquad \text{A - 7}$$

The results for a glass-epoxy resin composite are plotted in Fig. 9, for different values of the fibre volume fraction.

Elastic constants of a helical layer

A helical layer is composed of two interlocking half layers. In order to maintain continuous winding, it is necessary for the filaments to cross each other alternately at angles symmetrical to the longitudinal axis of the cylinder. The layer therefore is orthotropic and its principal directions coincide with the circumferential and longitudinal directions of the shell. Let I and II be the principal exes which coincide with x and y axes respectively. The strain components under the combined action of shear and direct stresses along these principal directions are given by

$$\begin{bmatrix} \epsilon_I \\ \epsilon_{II} \\ \epsilon_{I\,II} \end{bmatrix} = \begin{bmatrix} \frac{1}{E_I} & \frac{-\nu_{I\,II}}{E_{II}} & 0 \\ \frac{\nu_{II\,I}}{E_I} & \frac{1}{E_{II}} & \\ 0 & 0 & \frac{1}{G_{I\,II}} \end{bmatrix} \begin{bmatrix} \sigma_I \\ \sigma_{II} \\ \sigma_{I\,II} \end{bmatrix}$$

which can be written in the same form as equation (11), i.e.

$$[\epsilon] = [B][\sigma] \qquad \text{A - 8}$$

The strain system in the two half layers when expressed by equation (12) is

$$[\epsilon]_1 = [A]_1 [\sigma]_1$$

$$[\epsilon]_2 = [A]_2 [\sigma]_2$$

Where the subscripts 1 and 2 indicate half layer 1 and half layer 2. Since the two half layers are symmetrically placed about the principal axes, the transformed flexibility matrices show similarity. They are

$$[A]_1 = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}, \quad [A]_2 \begin{bmatrix} A_{11} & A_{12} & -A_{16} \\ A_{12} & A_{22} & -A_{26} \\ -A_{16} & -A_{26} & A_{66} \end{bmatrix}$$

The coefficients in terms of elastic constants and direction cosines are given in equation (14).

The equilibrium and strain compatability equations for the two half layers of the same thickness are

$$[\sigma]_1 + [\sigma]_2 = 2[\sigma]$$

and

$$[\varepsilon]_1 = [\varepsilon]_2 = [\varepsilon]$$

By substitution, we obtain

$$2\,[B]^{-1} = [A]_1^{-1} + [A]_2^{-1} \qquad \text{A - 9}$$

This gives the elastic constants along the principal axes I and II in terms of the elastic constants along the longitudinal and transverse directions of the fibres as given in equation 14.

$$E_I = \frac{A_{66}}{A_{11}A_{66} - A_{16}^2}$$

$$E_{II} = \frac{A_{66}}{A_{22}A_{66} - A_{26}^2}$$

$$\nu_{II\,I} = \frac{A_{16}A_{26} - A_{12}A_{66}}{A_{11}A_{66} - A_{16}^2} \qquad \text{A - 10}$$

$$G_{I\,II} = \frac{A_{11} A_{22} - A_{12}^{2}}{|A|}$$

Where $|A|$ is the determinant.

These elastic constants for a glass-epoxy resin composite are plotted in Fig. 10 against the orientation of the filaments.

Elastic constants of n orthotropic layers.

Filament wound cylindrical shells are composed of a number of orthotropic layers each of which by virtue of the winding process has its elastic principal axes coinciding with the longitudinal and circumferential directions of the shell. It is sometimes necessary to know its overall elastic constants along these directions. With the elastic constants of each layer already known, the determination of the overall elastic constant of the shell consisting of n layers can be carried out readily by applying the same approach as in the previous section. The flexibility matrix which relates the overall strain and stress conditions of the shell is

$$B = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{xy}}{E_y} & 0 \\ \frac{\nu_{yx}}{E_x} & \frac{1}{E_y} & 0 \\ 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix}$$

Applying the conditions of force equilibrium and strain compatability, equation A - 9 becomes

$$\left[B\right]^{-1} = \frac{1}{t} \sum_{i=1}^{n} t_i \left[B\right]_i^{-1} \qquad \text{A - 11}$$

The inverse flexibility matrices are in fact the stiffness matrices and equation (1) appears to be more convenient to apply. i.e.

$$\left[C\right] = \frac{1}{t} \sum_{i=1}^{n} t_i \left[C\right]_i = \frac{1}{t}\left[K\right] \qquad \text{A - 12}$$

and the coefficients of K are

$$K_{11} = \sum_{i=1}^{n} t_i \, (C_{11})_i$$

$$K_{12} = \sum_{i=1}^{n} t_i \, (C_{12})_i$$

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The stiffness matrix for the cylinder takes the following form with subscript i only to denote the ith layer.

$$\left[C\right] = \frac{1}{(1 - \nu_{xy}\nu_{yx})} \begin{bmatrix} E_x & \nu_{yx}E_y & 0 \\ \nu_{xy}E_x & E_y & 0 \\ 0 & 0 & G_{xy}(1-\nu_{xy}\nu_{y}) \end{bmatrix} \qquad A - 13$$

The overall elastic constants can now be obtained from equation A - 12. They are:

$$E_x = \frac{1}{t} \left(\frac{K_{11}K_{22} - K_{12}^2}{K_{22}} \right)$$

$$E_y = \frac{1}{t} \left(\frac{K_{11}K_{22} - K_{12}^2}{K_{11}} \right)$$

$$\nu_{xy} = \frac{K_{12}}{K_{22}} \qquad A - 14$$

$$\nu_{yx} = \frac{K_{12}}{K_{12}}.$$

$$G_{xy} = \frac{1}{t} K_{66}$$

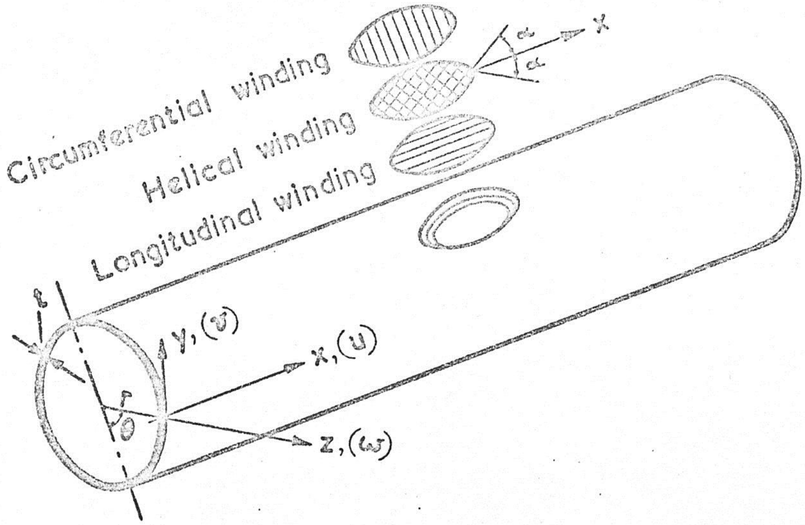

Figure 1. Filament-wound cylindrical shell.

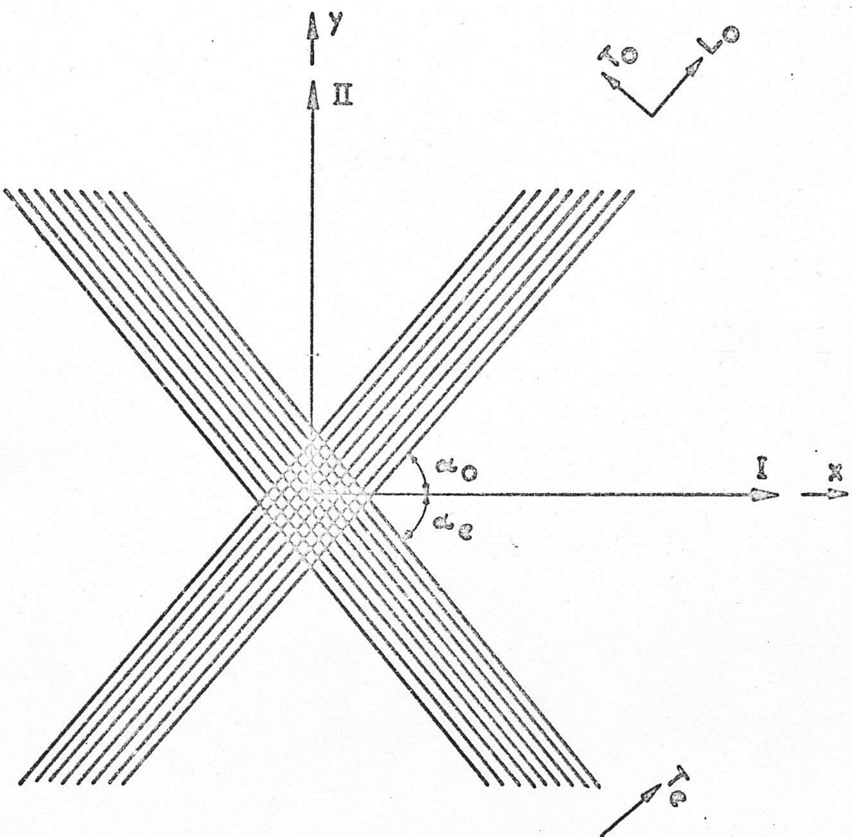

Figure 2. Orientation of filaments with reference to the principal axes.

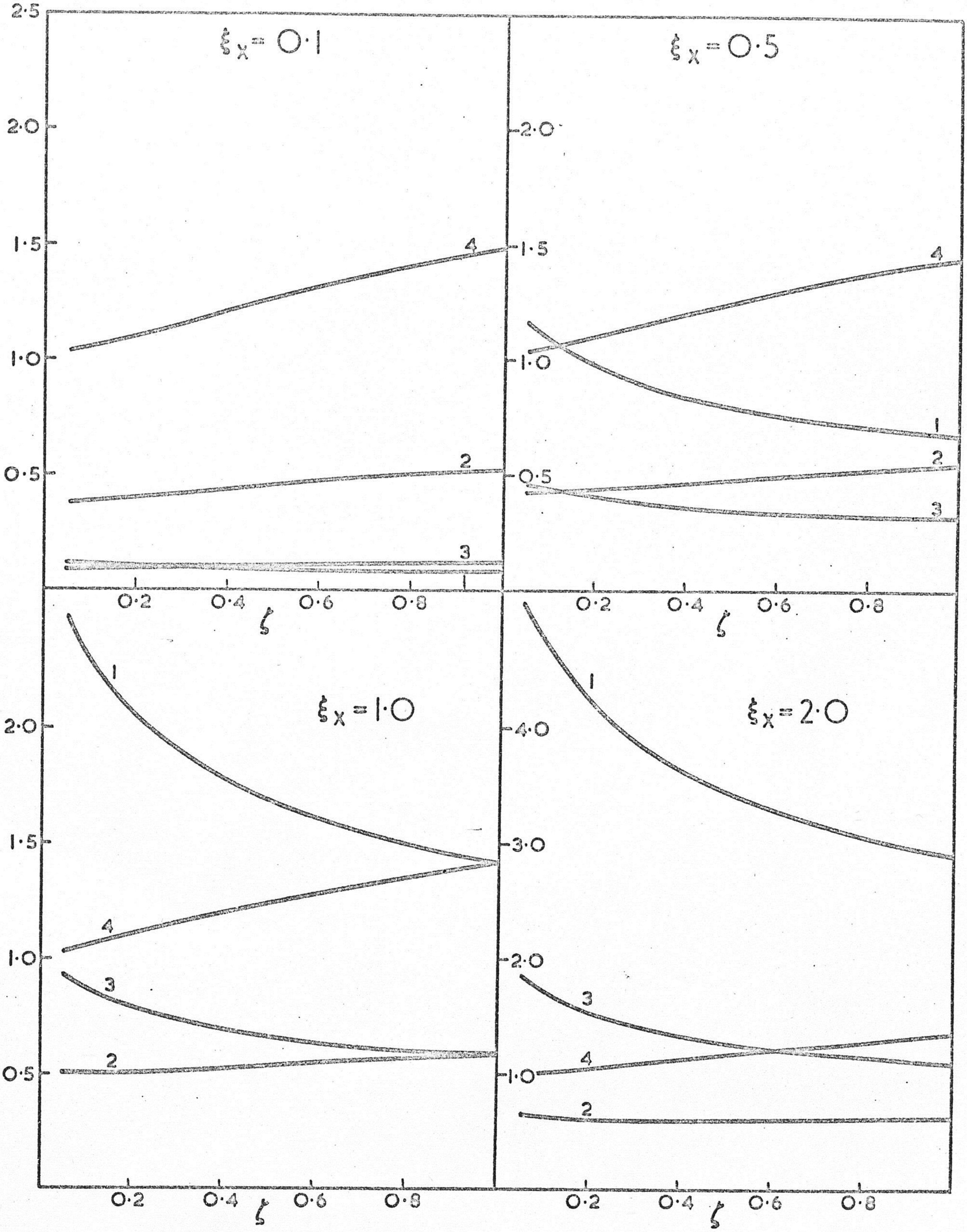

Figure 5. Nondimensional direct stress components of glass-epoxy resin cylinders with circumferential and longitudinal windings vs layer thickness ratio. curve 1 .. $(\Xi_x)_o$, curve 2 .. $(\Xi_\theta)_o$, curve 3 .. $(\Xi_x)_c$, curve 4 .. $(\Xi_\theta)_c$

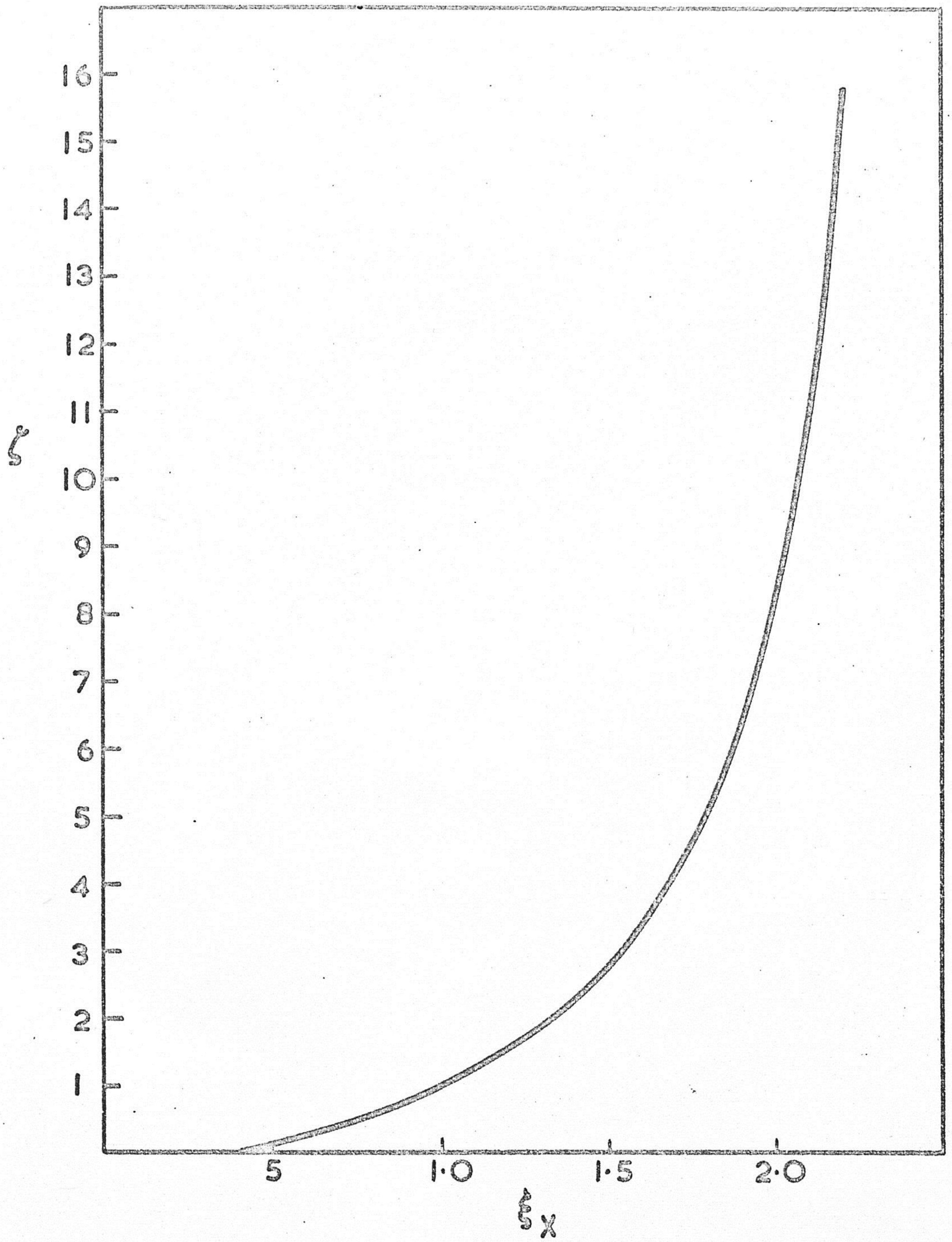

Figure 4. Optimum thickness ratio of cylinders with circumferential and longitudinal windings against applied direct stress ratio.

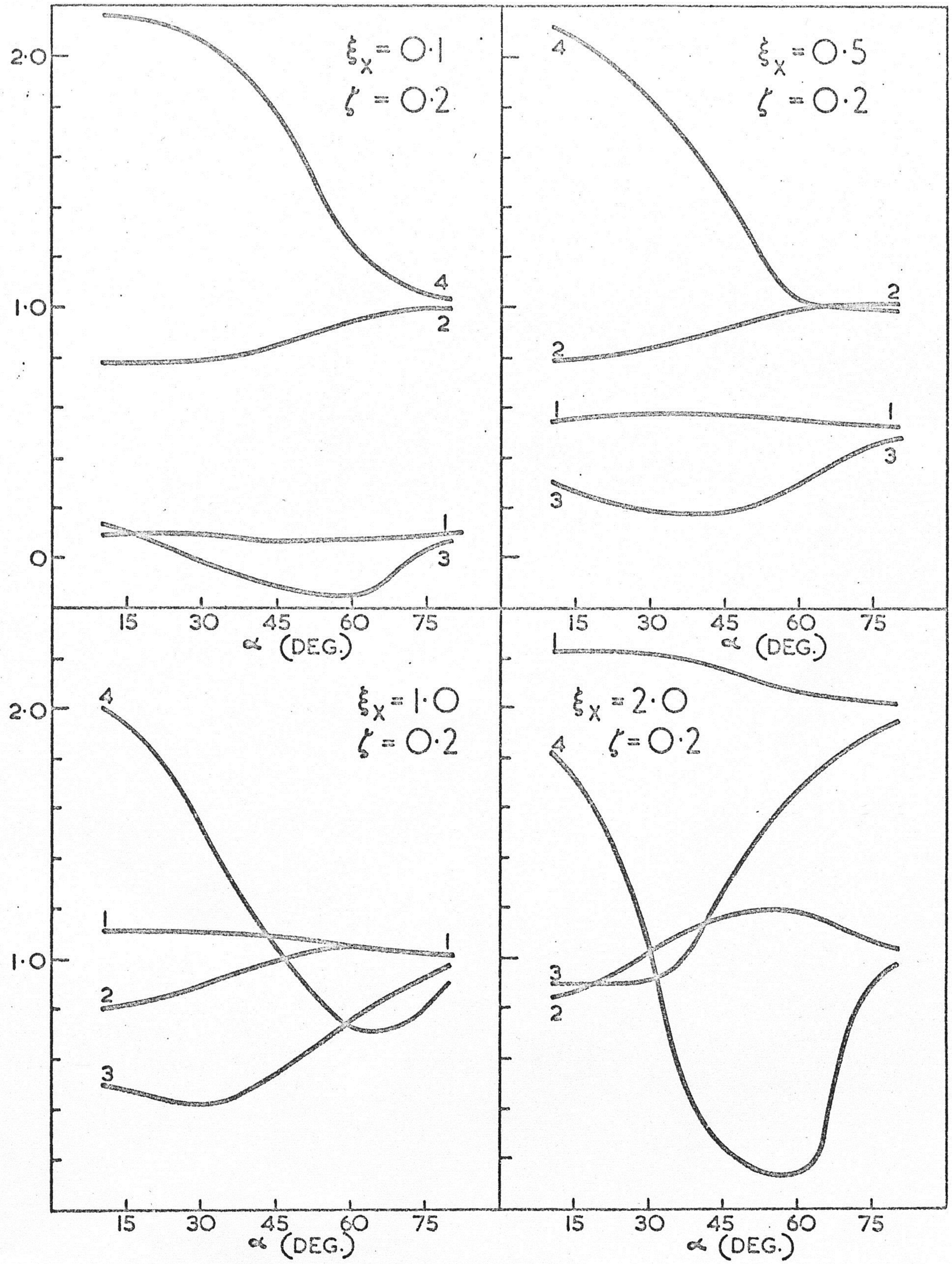

Figure 5. Nondimensional direct stress components of glass-epoxy resin cylinders with circumferential and helical windings against helical angle α. curve 1 ..$(\xi_x)_h$, curve 2 ..$(\xi_\theta)_h$, curve 3 ..$(\xi_x)_c$, curve 4 ..$(\xi_\theta)_c$.

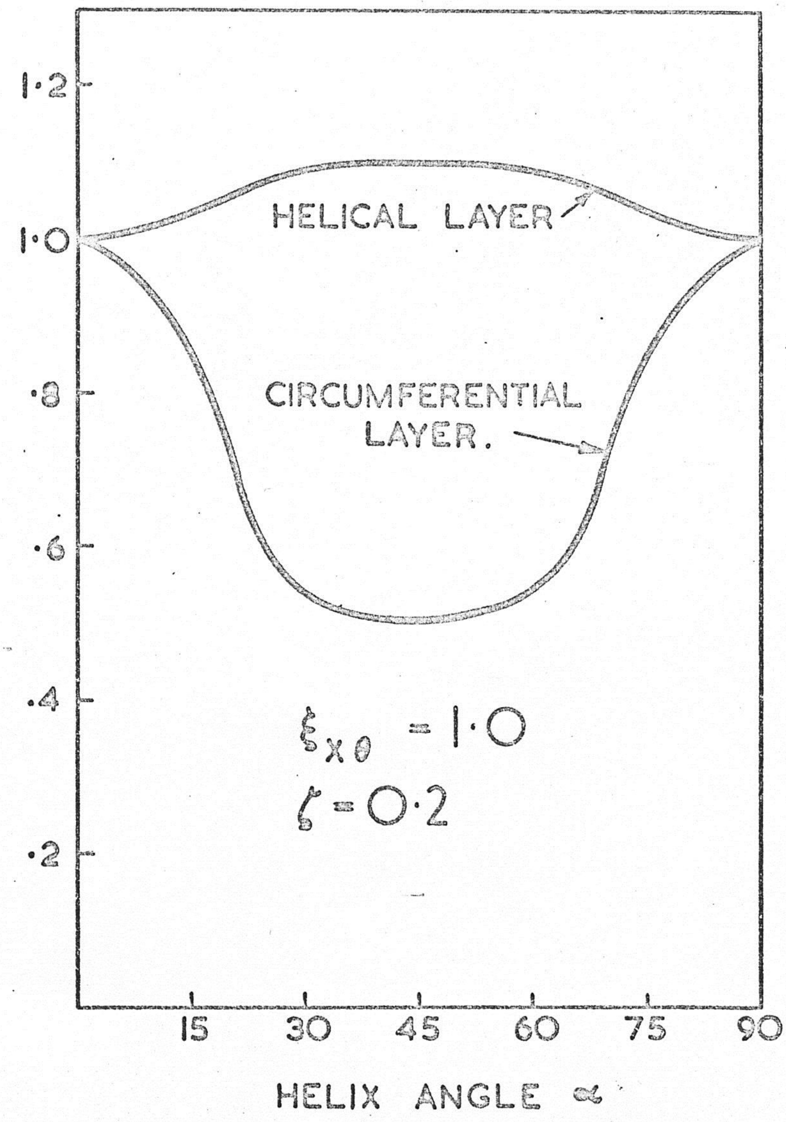

Figure 6. Nondimensional shear stress components of glass-epoxy resin cylinders with circumferential and helical windings against helical angle α.

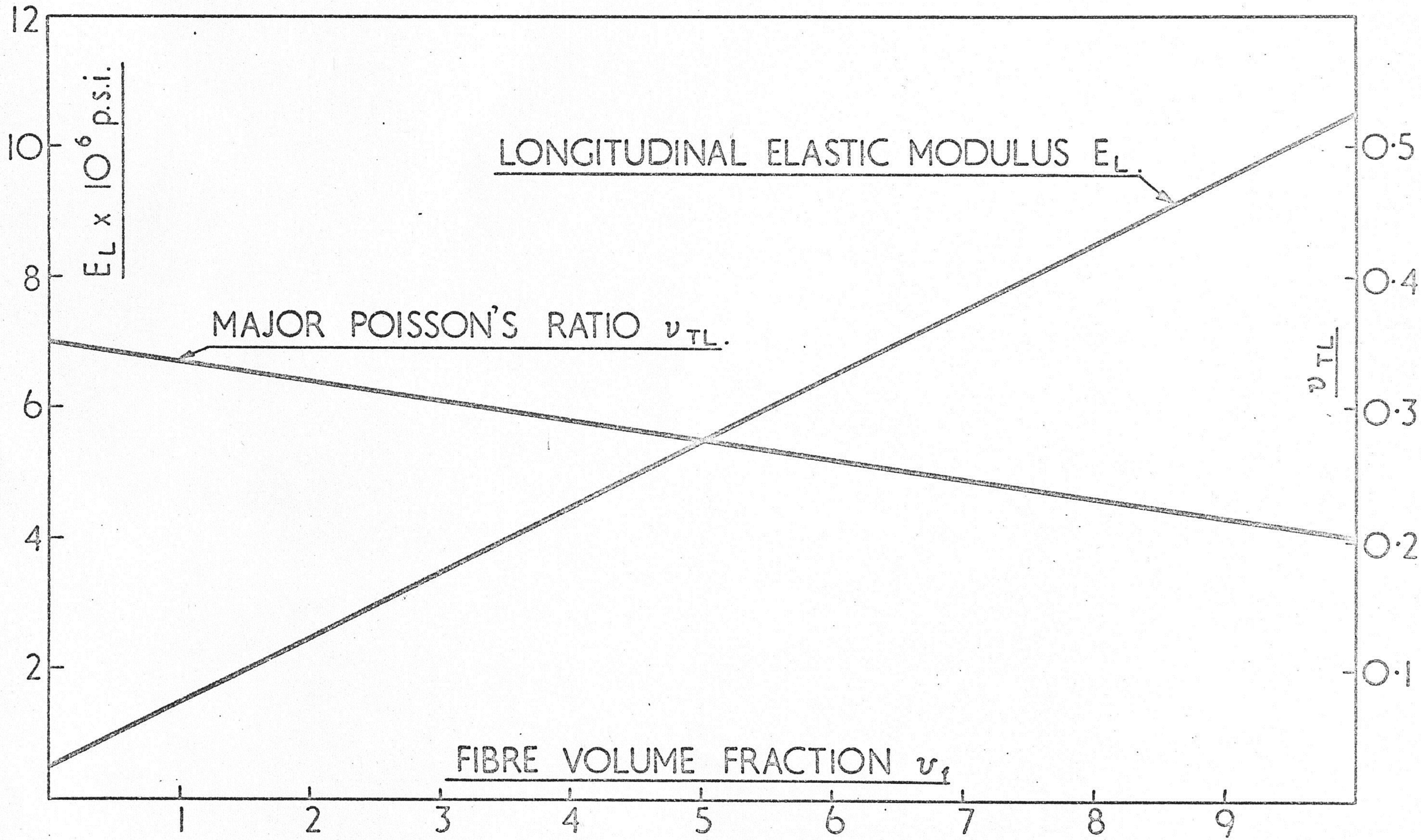

Figure 7. Longitudinal elastic modulus and the major Poisson's ratio of a glass-epoxy resin unidirectional layer against the fibre volume fraction.

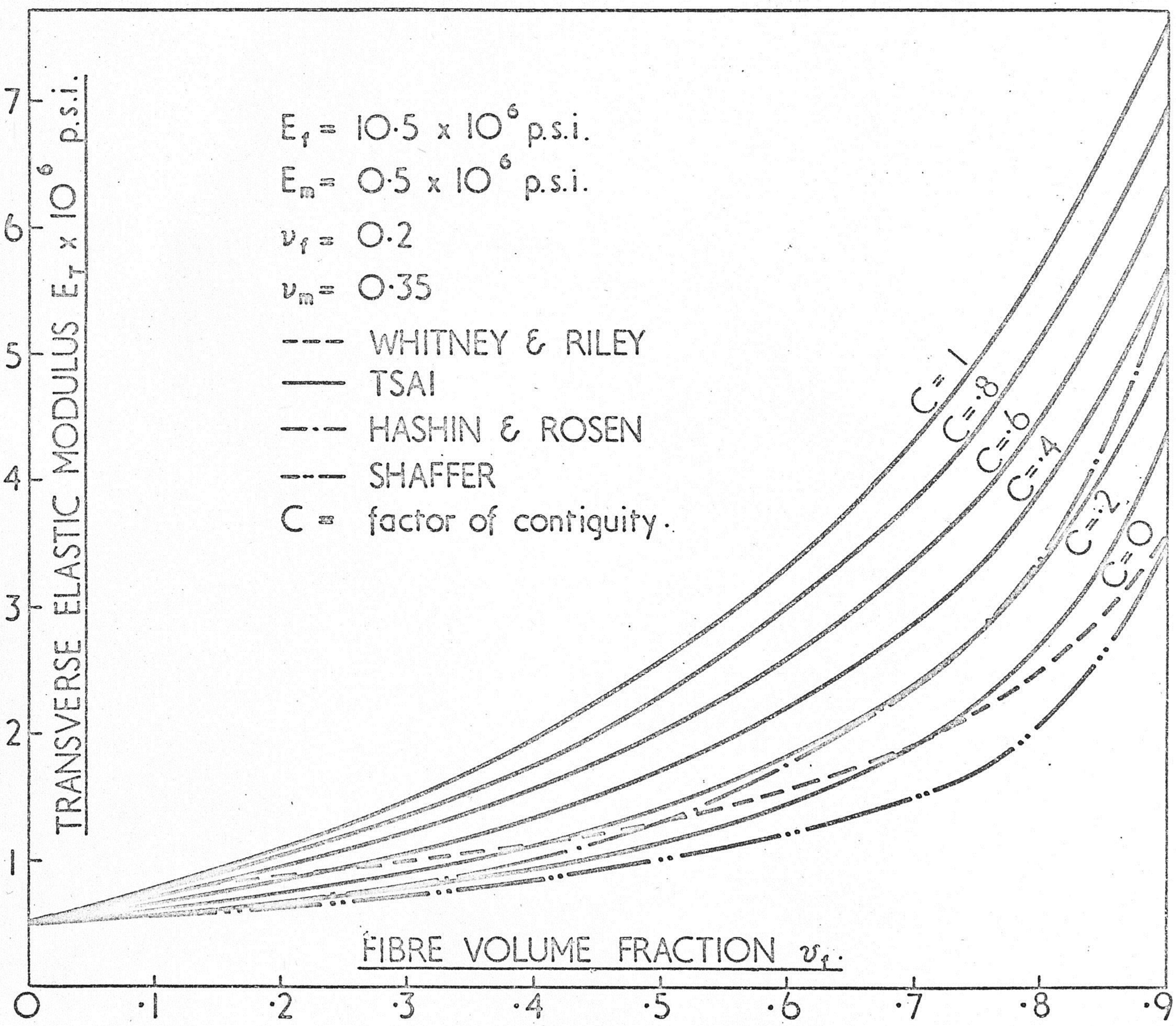

Figure 8. Transverse elastic modulus of a glass-epoxy unidirectional layer against the fibre volume fraction.

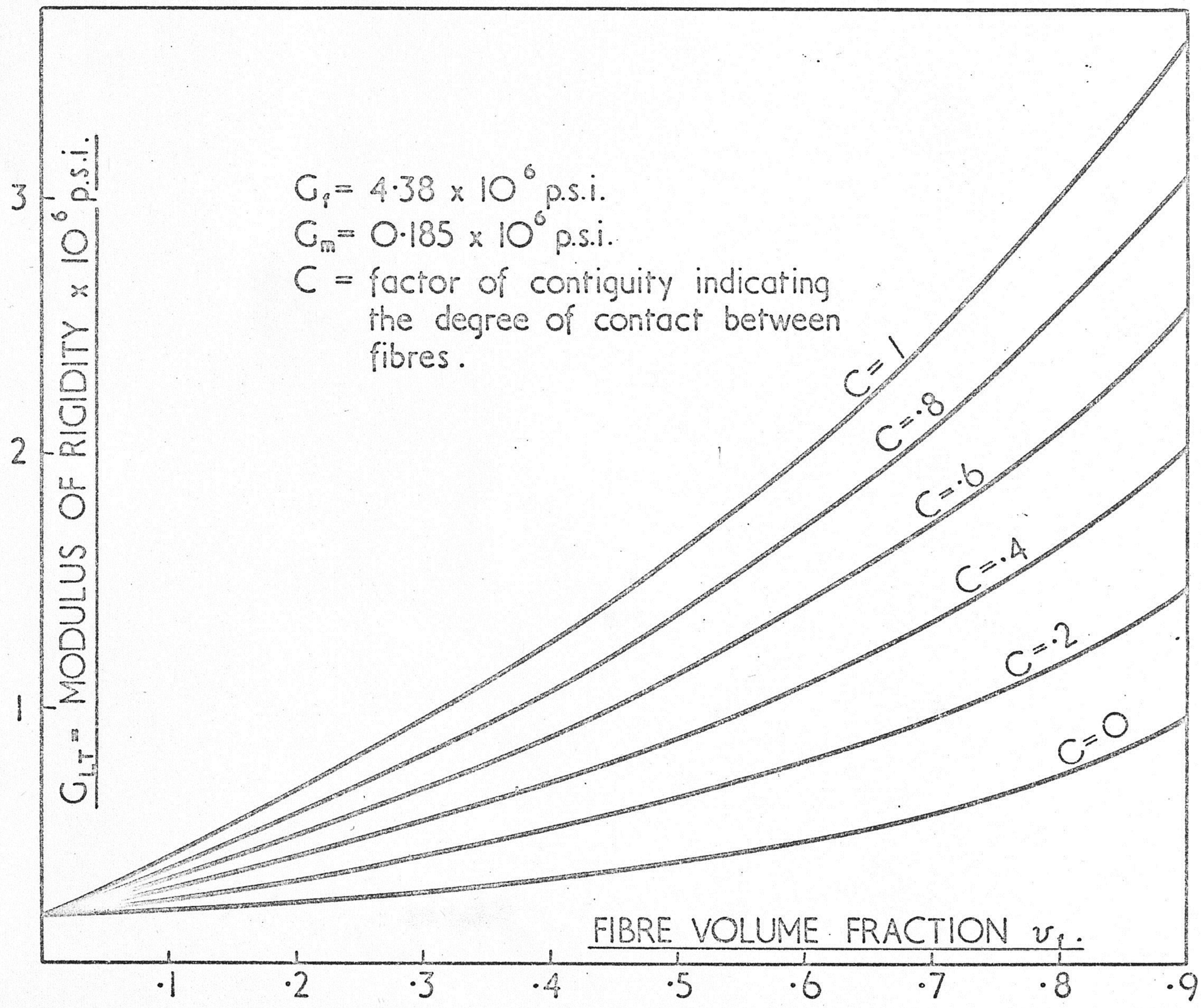

Figure 9. Shear modulus in the principal plane of a glass-epoxy resin unidirectional layer against the fibre volume fraction.

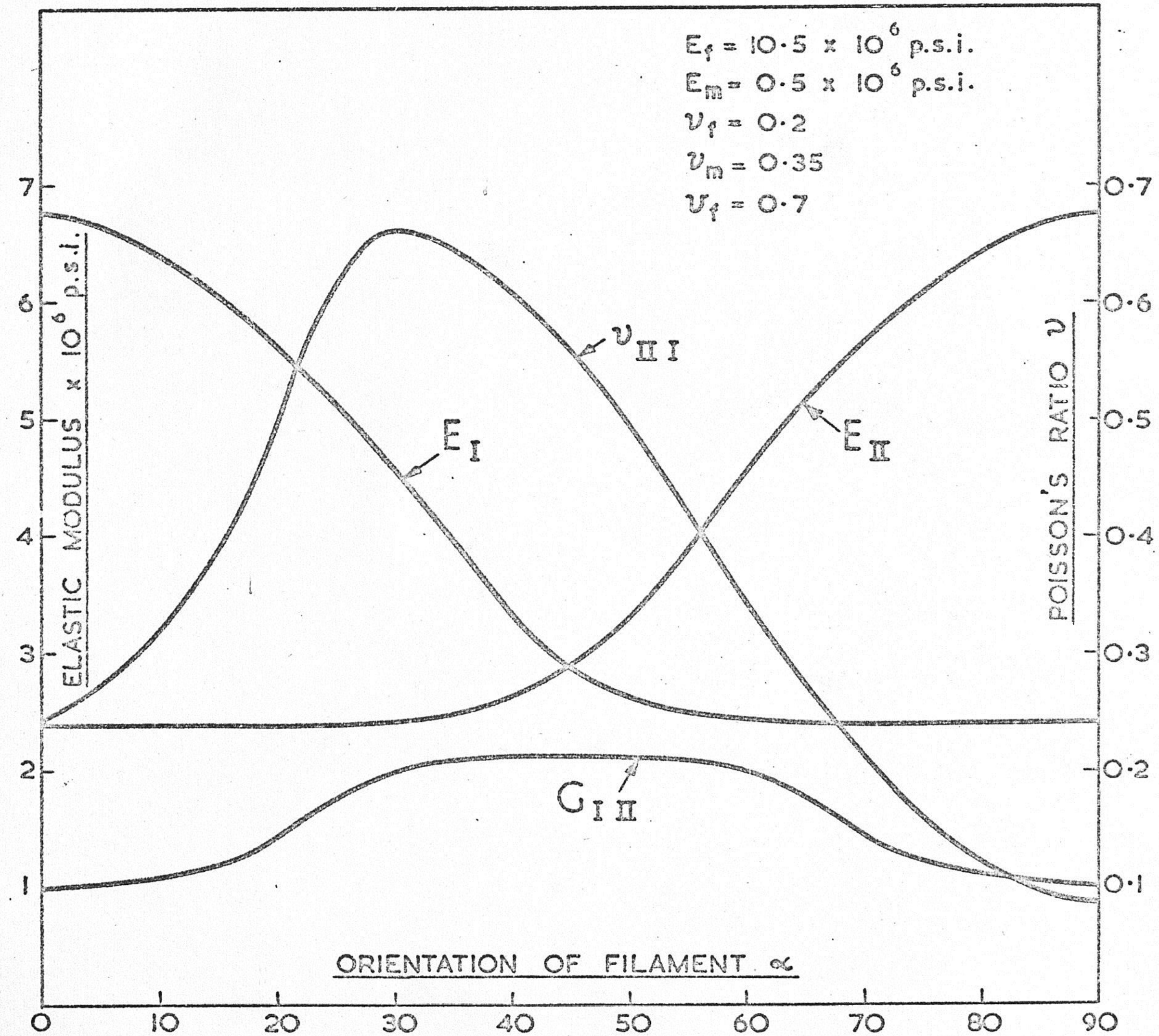

Figure 10. Elastic constants along the principal directions of a glass-epoxy resin helical layer against the helical angle α.